Approximate green function of the linear Boltzmann equation for finite media (J. DEVOOGHT, ULB)

1. Introduction

2. The natural tensor interpolant of Thomas

Let $f(x,y)$ be a bivariate function, and $\{L_i\}_{i=1}^{m}$, $\{K_j\}_{j=1}^{n}$ two set of linear functionals which operate respectively on x and y and which are commutative : $L_i K_j f = K_j L_i f$

Let F be the $m \times n$ matrix of element $F_{ij} = L_i K_j f$ and F^+ its generalized Moore-Penrose inverse, ie the unique matrix ~~which~~ which satisfies the four conditions

$$
\begin{array}{ll}
(i) & FF^+F = F \\
(ii) & F^+FF^+ = F^+ \\
(iii) & (FF^+)^T = FF^+ \\
(iv) & (F^+F)^T = F^+F
\end{array}
\tag{2.1}
$$

We define the vectors

$$
\begin{aligned}
\overline{K} &= (K_1 f \dots K_n f) \\
\overline{L} &= (L_1 f \dots L_m f)
\end{aligned}
\tag{2.2}
$$

The function

$$T(f) = t(x,y) = \vec{K} F^+ \vec{L}^T \tag{2.3}$$

is the natural tensor interpolant of Thomas

It satisfies the idempotency relation

$$T(T(f)) = T(f) \tag{2.4}$$

In general T is a non linear operator, ie

$$T(f+g) \neq T(f) + T(g) \tag{2.5}$$

It reproduces every tensor product

For independent linear functionals, F is not always be full rank, but if it is then $FF^+ = I_m$, the identity matrix of rank $m \leq n$

and

$$
\begin{aligned}
L_i T(f) &= L_i \sum_{j,k} K_j f (F^+)_{jk} (L_k f) = \sum_{jk} (FF^+)_{ik} L_k f \\
&= L_i f
\end{aligned}
\tag{2.6}
$$

Similarly if it's full rank $n \leq m$

$$K_i T(f) = K_i f \qquad (2.7)$$

For the square case $n = m$ and F non singular, $F^+ = F^{-1}$, $T(f)$ satisfies both interpolation conditions (2.6)(2.7)

3. Construction of an approximate green function

The Green function $G(\bar{r}, \bar{\Omega} | \bar{r}_0, \bar{\Omega}_0)$ for the Boltzmann equation with vacuum boundary condition is a "bivariate" function of $(\bar{r}, \bar{\Omega})$ and $(\bar{r}_0, \bar{\Omega}_0)$

We write

$$G(\bar{r}, \bar{\Omega} | \bar{r}_0, \bar{\Omega}_0) = G_\infty(\bar{r}, \bar{\Omega} | \bar{r}_0, \bar{\Omega}_0) + g(\bar{r}, \bar{\Omega} | \bar{r}_0, \bar{\Omega}_0) \qquad (3.1)$$

where G_∞ is the infinite medium green function. Equation (3.1) defines g. However the fundamental singularity of G is taken care of by G_∞ and g is a relatively smooth function far from the boundaries. It is easier therefore to approximate g than G inasmuch as G_∞ is explicitly known

From the boundary conditions: $\quad G(\bar{r}_s, \bar{\Omega} | \bar{r}_0, \bar{\Omega}_0) = 0 \quad , r_s \in \partial D, \ \bar{n}_s \bar{\Omega} < 0$

we have

$$g(\bar{r}_s, \bar{\Omega} | \bar{r}_0, \bar{\Omega}_0) = - G_\infty(\bar{r}_s, \bar{\Omega} | \bar{r}_0, \bar{\Omega}_0) \quad , \forall r_s \in \partial D, \ \bar{n}_s \bar{\Omega} < 0 \qquad (3.2)$$

which is a transfinite interpolation condition. If we take the same linear functionals of both members, say

$$L_i g(\bar{r}, \bar{\Omega} | \bar{r}_0, \bar{\Omega}_0) = - L_i G_\infty(\bar{r}, \bar{\Omega} | \bar{r}_0, \bar{\Omega}_0) \quad , i = 1 \dots N \qquad (3.3)$$

with

$$\boxed{L_i f(\bar{r}, \bar{\Omega}) \triangleq \int_\Gamma d\bar{r}_s \int_{\bar{n}_s \bar{\Omega} < 0} |\bar{n}_s \bar{\Omega}| \varphi_i^*(\bar{r}_s, \bar{\Omega}) f(\bar{r}_s, \bar{\Omega}) \, d\bar{\Omega}} \qquad (3.4)$$

Any g which satisfies (3.2) satisfies also (3.3) but the reciprocal is not true

We define now the Thomas interpolant

$$\hat{g}(\bar{r},\bar{\Omega}|\bar{r}_0,\bar{\Omega}_0) = \sum_i \sum_j L_i g(\bar{r}_0,\bar{\Omega}_0)[g^+]_{ij} K_j g(\bar{r},\bar{\Omega}) \qquad (3.5)$$

with $\quad [g^+]_{ij} = [L_i K_j g]^+ \qquad (3.6)$

Since L_i operates on $\bar{r},\bar{\Omega}$, the functional $L_i g$ is a function of $\bar{r}_0,\bar{\Omega}_0$ which we write $L_i g(\bar{r}_0,\bar{\Omega}_0)$. Functional K_j is so far undefined

We have the interpolation $\quad \hat{g} = Tg \quad$ with $\quad T(Tg) = \hat{g} \qquad (3.7)$

Since we have no reason to discriminate between $(\bar{r},\bar{\Omega})$ and $(\bar{r}_0,\bar{\Omega}_0)$, $m=n$ and assuming $[L_i K_j g]$ non singular

$$\left.\begin{aligned} L_i\hat{g} &= L_i g \\ K_j\hat{g} &= K_j g \end{aligned}\right\} \qquad (3.8)$$

We define $\quad \hat{G}(\bar{r},\bar{\Omega}|\bar{r}_0,\bar{\Omega}_0) = G_\infty(\bar{r},\bar{\Omega}|\bar{r}_0,\bar{\Omega}_0) + \hat{g}(\bar{r},\bar{\Omega}|\bar{r}_0,\bar{\Omega}_0) \qquad (3.9)$

Then $\quad L_k\hat{G} = L_k G_\infty + L_k\hat{g} = L_k G_\infty + L_k g = 0 \qquad (3.10)$

and the approximate Green function defined by (3.9) satisfies the reduced set of boundary conditions (3.8)

So far $\hat{G}$ does not approximate G in any sense except the boundary conditions, but does it satisfy the Boltzmann equation?

The theorem of reciprocity is verified if

$$\hat{g}(\bar{r},\bar{\Omega}\,|\bar{r}_0,\bar{\Omega}_0) = \hat{g}(\bar{r}_0,-\bar{\Omega}_0\,|\bar{r},-\bar{\Omega}) \qquad (3.11)$$

or $$\sum_{ij} L_i g(\bar{r}_0,\bar{\Omega}_0)\,[g^+]_{ij}\,K_j g(\bar{r},\bar{\Omega}) = \sum_{ij} L_i g(\bar{r},-\bar{\Omega})\,[g^+]_{ij}\,K_j g(\bar{r}_0,-\bar{\Omega}_0)$$

i.e. if (a) $$L_i g(\bar{r}_0,\bar{\Omega}_0) = K_i g(\bar{r}_0,-\bar{\Omega}_0) \qquad (3.12)$$

and (b) $[g^+]_{ij}$ is symmetric

From (3.12) (3.4)

$$K_i g(\bar{r}_0,-\bar{\Omega}_0) = L_i g(\bar{r}_0,\bar{\Omega}_0) = \int_\Gamma d\bar{r}\int_{\bar{n}\bar{\Omega}<0} d\bar{\Omega}\,|\bar{n}\bar{\Omega}|\,\varphi_i^*(\bar{r},\bar{\Omega})\,g(\bar{r},\bar{\Omega}\,|\bar{r}_0,\bar{\Omega}_0)$$

$$= \int_\Gamma d\bar{r}\int_{\bar{n}\bar{\Omega}<0} d\bar{\Omega}\,|\bar{n}\bar{\Omega}|\,\varphi_i^*(\bar{r},\bar{\Omega})\,g(\bar{r}_0,-\bar{\Omega}_0\,|\bar{r},-\bar{\Omega})$$

$$K_i g(\bar{r}_0,-\bar{\Omega}_0) = \int_\Gamma d\bar{r}\int_{\bar{n}\bar{\Omega}>0} d\bar{\Omega}\,|\bar{n}\bar{\Omega}|\,\varphi_i^*(\bar{r},-\bar{\Omega})\,g(\bar{r}_0,-\bar{\Omega}_0\,|\bar{r},\bar{\Omega})$$

$$\boxed{K_i g(\bar{r},+\bar{\Omega}) = \int_\Gamma d\bar{r}_0\int_{\bar{n}\bar{\Omega}_0>0} d\bar{\Omega}_0\,|\bar{n}\bar{\Omega}_0|\,\varphi_i^*(\bar{r}_0,-\bar{\Omega}_0)\,g(\bar{r},+\bar{\Omega}\,|\bar{r}_0,\bar{\Omega}_0)} \qquad (3.13)$$

~~which gives the definition of K_i, which was so far undefined~~

Then $$L_j K_i g(\bar{r},\bar{\Omega}) = \int_\Gamma d\bar{r}\int_{\bar{n}\bar{\Omega}<0} d\bar{\Omega}\,|\bar{n}\bar{\Omega}|\,\varphi_j^*(\bar{r},\bar{\Omega})\int_\Gamma d\bar{r}_0\int_{\bar{n}\bar{\Omega}_0>0}|\bar{n}\bar{\Omega}_0|\,d\bar{\Omega}_0\,\varphi_i^*(\bar{r}_0,-\bar{\Omega}_0)\,g(\bar{r},\bar{\Omega}\,|\bar{r}_0,\bar{\Omega}_0)$$

$$= \int_\Gamma d\bar{r}_0\int_{\bar{n}\bar{\Omega}_0>0} d\bar{\Omega}_0\,|\bar{n}\bar{\Omega}_0|\,\varphi_i^*(\bar{r}_0,-\bar{\Omega}_0)\int_\Gamma d\bar{r}\int_{\bar{n}\bar{\Omega}<0}|\bar{n}\bar{\Omega}|\,\varphi_j^*(\bar{r},\bar{\Omega})\,g(\bar{r}_0,-\bar{\Omega}_0\,|\bar{r},-\bar{\Omega})\,d\bar{\Omega}$$

If we change $\bar{\Omega},\bar{\Omega}_0 \to -\bar{\Omega},-\bar{\Omega}_0$

$$L_j K_i g(\bar{r},\bar{\Omega}) = L_i K_j g(\bar{r},\bar{\Omega}) \qquad (3.14)$$

which entails that $[g^+]_{ij}$ is symmetric

Therefore the choice of K_i guarantees that $\hat{g}$ satisfies the theorem of reciprocity

Finally:

$$\boxed{\hat{G}(\bar{r},\bar{\Omega}\,|\bar{r}_0,\bar{\Omega}_0) = G_\infty(\bar{r},\bar{\Omega}\,|\bar{r}_0,\bar{\Omega}_0) - \sum_i\sum_j L_i G_\infty(\bar{r}_0,\bar{\Omega}_0)\left[L_i K_j G_\infty\right]^+ K_j G_\infty(\bar{r},\bar{\Omega})} \qquad (3.15)$$

$$K_j g_\infty(\bar r,\bar\Omega) = \int_\Gamma d\bar r_0 \int_{\bar n\bar\Omega>0} G_\infty(\bar r,\bar\Omega|\bar r_0,\bar\Omega_0)\,|\bar n\bar\Omega_0|\,\varphi_i^*(\bar r_0,-\bar\Omega_0)\,d\bar\Omega_0 \qquad (3.16)$$

$$\mathcal{T} K_j G_\infty(\bar r,\bar\Omega) = K_j\,\mathcal{T} G_\infty(\bar r,\bar\Omega) = \int_\Gamma d\bar r_0 \int_{\bar n\bar\Omega>0} \delta(\bar r-\bar r_0)\,\delta(\bar\Omega-\bar\Omega_0)\,|\bar n\bar\Omega_0|\,\varphi_i^*(\bar r_0,-\bar\Omega_0)$$

$$= 0 \quad \text{for } \bar r \text{ inside } \partial D \qquad (3.17)$$

Then $\mathcal{T}\hat G = \mathcal{T} G_\infty = \delta(\bar r-\bar r_0)\,\delta(\bar\Omega-\bar\Omega_0)$ (3.18)

and $\hat G$ satisfies Boltzmann equation

In conclusion the approximate green function (i) satisfies Boltzmann equation, (ii) the reciprocity theorem, (iii) the approximate boundary condition

$$\int_\Gamma d\bar r_s \int_{\bar n\bar\Omega<0} |\bar n\bar\Omega|\,\varphi_i^*(\bar r,\bar\Omega)\,G(\bar r,\bar\Omega|\bar r_0,\bar\Omega_0)\,d\bar\Omega = 0, \quad i=1\dots N \qquad (3.19)$$

and therefore : $\mathcal{T}\hat G(\bar r,\bar\Omega|\bar r_0,\bar\Omega_0) = \delta(\bar r-\bar r_0)\,\delta(\bar\Omega-\bar\Omega_0)$

if $\mathcal{T} K_j g(\bar r,\bar\Omega) = 0$ (3.15bis)

Let us define

$$\tilde{\hat{g}}(\bar r,\bar\Omega|\bar r_0,\bar\Omega_0) \equiv \sum_n\sum_m \mathcal{S}_n(\bar r,\bar\Omega)[A^{-1}]_{nm}\,\mathcal{S}_m^{*}(\bar r_0,\bar\Omega_0) \qquad (3.20)$$

Then $$L_iK_j\,\tilde{\hat{g}} = \sum_n\sum_m [L_i\mathcal{S}_n(\bar r,\bar\Omega)][K_j\mathcal{S}_m^{*}(\bar r_0,\bar\Omega_0)][A^{-1}]_{nm} \qquad (3.21)$$

We have $$K_jL_i\,\tilde{\hat{g}}(\bar r,\bar\Omega|\bar r_0,\bar\Omega_0) = K_jL_i\,\hat g(\bar r,\bar\Omega|\bar r_0,\bar\Omega_0) \qquad (3.22)$$

if $$\sum_{n,m}[L_i\mathcal{S}_n(\bar r,\bar\Omega)][A^{-1}]_{nm}[K_j\mathcal{S}_m^{*}(\bar r_0,\bar\Omega_0)] = K_jL_i\hat g = -K_jL_iG_\infty$$

We define (3.23) $$\boxed{C_{ij} = L_i\mathcal{S}_j(\bar r,\bar\Omega) = \int d\bar r\int_{\bar n\bar\Omega<0} |\bar n\bar\Omega|\,\mathcal{S}_i^{*}(\bar r,\bar\Omega)\,\mathcal{S}_j(\bar r,\bar\Omega)\,d\bar\Omega}$$

(3.24) $$\boxed{D_{ji} = K_i\mathcal{S}_j^{*}(\bar r_0,\bar\Omega_0) = \int_\Gamma d\bar r_0\int_{\bar n\bar\Omega_0>0} |\bar n\bar\Omega_0|\,\mathcal{S}_i^{*}(\bar r_0,\bar\Omega_0)\,\mathcal{S}_j^{*}(\bar r_0,\bar\Omega_0)\,d\bar\Omega}$$

Then $$(CA^{-1}D)_{ij} = L_iK_j\,\hat g \qquad (3.25)$$

or $$[A^{-1}]_{ij} = C^{-1}\hat G D^{-1} \qquad (3.26)$$

where $\hat G$ is the matrix of element $\hat G_{ij} = L_iK_j\hat g = -L_iK_jG_\infty$ (3.27)

It has not escaped to the attention of the reader that (3.5) is useful insofar as $g(\bar r,\bar\Omega|\bar r_0,\bar\Omega_0)$ is known, which amounts to assume the problem solved.

We substitute to $\hat g(\bar r,\bar\Omega|\bar r_0,\bar\Omega_0)$ function $\tilde{\hat g}(\bar r,\bar\Omega|\bar r_0,\bar\Omega_0)$

What properties does $\tilde{\hat g}$ share with $\hat g$ besides (3.22)?

For all incoming fluxes which are linear combinations of $\mathcal{S}_i(\bar r,\bar\Omega)\Big|_{i=1}^{N}$ the angular flux calculated from

$$\int_\Gamma d\bar r_0\int_{\bar n\bar\Omega_0<0}\hat G(\bar r,\bar\Omega|\bar r_0,\bar\Omega_0)\sum_k a_k\mathcal{S}_k(\bar r_0,\bar\Omega_0)\,|\bar n\bar\Omega_0|\,d\bar\Omega_0$$

is preferred if

$$L_i \int_\Gamma d\bar{r}_0 \int_{\bar{n}\bar{\Omega}_0<0} \hat{g}(\bar{r},\bar{\Omega}|\bar{r}_0,\bar{\Omega}_0)\,|\bar{n}\bar{\Omega}_0|\,\mathcal{S}_k(\bar{r}_0,\bar{\Omega}_0)\,d\bar{\Omega}_0 = L_i \int_\Gamma d\bar{r}_0 \int_{\bar{n}\bar{\Omega}_0<0} \hat{\tilde{g}}(\bar{r},\bar{\Omega}|\bar{r}_0,\bar{\Omega}_0)\,|\bar{n}\bar{\Omega}_0|\,\mathcal{S}_k(\bar{r}_0,\bar{\Omega}_0) \qquad (3.28)$$

The theorem of reciprocity is verified if

$$\sum_{n,m} \mathcal{S}_n(\bar{r},\bar{\Omega})[A^{-1}]_{nm}\,\mathcal{S}_m^*(\bar{r}_0,\bar{\Omega}_0) = \sum_{n,m} \mathcal{S}_n^*(\bar{r},-\bar{\Omega})[A^{-1}]_{mn}\,\mathcal{S}_m(\bar{r}_0,-\bar{\Omega}_0)$$

i.e A is symmetric and $\mathcal{S}_m^*(\bar{r}_0,\bar{\Omega}_0) = \mathcal{S}_m(\bar{r}_0,-\bar{\Omega}_0)$ (3.29)

The last condition guarantees that A is symmetric because (3.24) gives

$$D_{ji} = \int_\Gamma d\bar{r}_0 \int_{\bar{n}\bar{\Omega}_0>0} |\bar{n}\bar{\Omega}_0|\,\mathcal{S}_i^*(\bar{r}_0,-\bar{\Omega}_0)\,\mathcal{S}_j^*(\bar{r}_0,\bar{\Omega}_0)\,d\bar{\Omega}_0$$

$$= \int_\Gamma d\bar{r} \int_{\bar{n}\bar{\Omega}<0} |\bar{n}\bar{\Omega}|\,\mathcal{S}_i^*(\bar{r},\bar{\Omega})\,\mathcal{S}_j^*(\bar{r},-\bar{\Omega})\,d\bar{\Omega} = \mathbf{C}_{ij}$$

ie $D = C^T$ and $A^{-1} = C^{-1}\hat{G}C^{-T} = C^{-1}\hat{G}^T C^{-T} = A^{-T}$

Having determined $\mathcal{S}_m^*$ in terms of $\mathcal{S}_m$, we fulfill condition (3.19bi) having $\mathcal{T}\tilde{\hat{g}} = 0$, i.e $\mathcal{T}\mathcal{S}_n(\bar{r},\bar{\Omega}) = 0$

Therefore the $\mathcal{S}_n(\bar{r},\bar{\Omega})$ are solutions of the homogeneous Boltzmann equations

For homogeneous medium characterized by $c<1$ $\mathcal{S}_n(\bar{r},\bar{\Omega})$ is

$$\mathcal{S}_n(\bar{r},\bar{\Omega}) = \frac{e^{\bar{\nu}_m\bar{r}}}{1+\frac{\bar{\nu}\bar{\Omega}}{\sigma}} \quad (3.30), \text{ or linear combinations thereof}$$

where $|\bar{\nu}_m| = \nu$ is solution of

$$1 = c\,\frac{\operatorname{arctanh}\nu/\sigma}{\nu/\sigma} \qquad (3.31)$$

In conclusion, the algorithm is the following:

- starting from a set (3.30) and the knowledge of G_∞ we compute C, D from (3.23)(3.24), next $\hat{G}$ from (3.27) and finally A^{-1} from (3.26) which gives $\hat{\tilde{g}}$ from (3.20)

4. Response matrix formalism

The solution of a problem with distributed source $Q(\vec r,\vec\Omega)$ and incident angular flux is

$$\phi(\vec r,\vec\Omega) = \int_{\mathcal D} d\vec r_0 \int_{4\pi} G(\vec r,\vec\Omega|\vec r_0,\vec\Omega_0)\,Q(\vec r_0,\vec\Omega_0)\,d\vec\Omega_0 + \int_\Gamma d\vec r_0 \int_{\vec n\vec\Omega_0<0} |\vec n\vec\Omega_0|\,G(\vec r,\vec\Omega|\vec r_0,\vec\Omega_0)\,\psi_{inc}(\vec r_0,\vec\Omega_0)\,d\vec\Omega_0$$

We expand

$$\psi_{inc}(\vec r,\vec\Omega) = \sum_n a_n \mathcal{S}_n(\vec r,\vec\Omega) \qquad \vec r\in\partial\mathcal D$$
$$Q(\vec r,\vec\Omega) = \sum_n q_n \mathcal{S}_n(\vec r,\vec\Omega)$$

$$L_i\psi_{inc}(\vec r,\vec\Omega) = \sum_n a_n \int_\Gamma d\vec r_s \int_{\vec n_\rho\vec\Omega_0<0} |\vec n\vec\Omega_0|\,\mathcal{S}_i^*(\vec r_s,\vec\Omega_0)\,\mathcal{S}_n(\vec r_0,\vec\Omega_0)\,d\vec\Omega_0$$
$$= \sum_n a_n C_{in} = (C\vec a)_i$$

i.e.

$$a_n = \sum_i (C^{-1})_{ni} L_i\psi_{inc}$$

$$\phi(\vec r,\vec\Omega) = \sum_n q_n \int_{\mathcal D} d\vec r_0 \int_{4\pi} G(\vec r,\vec\Omega|\vec r_0,\vec\Omega_0)\,\mathcal{S}_n(\vec r_0,\vec\Omega_0)\,d\vec\Omega_0$$
$$+ \int_\Gamma d\vec r_0 \int_{\vec n\vec\Omega_0<0} |\vec n\vec\Omega_0|\,G(\vec r,\vec\Omega|\vec r_0,\vec\Omega_0) \sum_n \mathcal{S}_n(\vec r_0,\vec\Omega_0) \sum_i (C^{-1})_{ni} L_i\psi_{inc}$$

Define

$$\boxed{\begin{aligned} L_i^* &= \int_\Gamma d\vec r_s \int_{\vec n\vec\Omega>0} |\vec n\vec\Omega|\,\mathcal{S}_i^*(\vec r_s,\vec\Omega)\cdots d\vec\Omega \\ K_i^* &= \int_\Gamma d\vec r_0 \int_{\vec n\vec\Omega_0<0} |\vec n\vec\Omega_0|\,\mathcal{S}_i^*(\vec r_0,-\vec\Omega_0)\cdots d\vec\Omega_0 \end{aligned}}$$

Then

$$L_i^*\phi(\vec r,\vec\Omega) = \sum_n q_n \iint L_i^* G(\vec r,\vec\Omega|\vec r_0,\vec\Omega_0)\,\mathcal{S}_n(\vec r_0,\vec\Omega_0)\,d\vec r_0\,d\vec\Omega_0$$
$$+ \sum_n \int_\Gamma d\vec r_0 \int_{\vec n\vec\Omega_0<0} |\vec n\vec\Omega_0|\,\mathcal{S}_n(\vec r_0,\vec\Omega_0)\,L_i^* G(\vec r,\vec\Omega|\vec r_0,\vec\Omega_0) \sum_j (C^{-1})_{mj} L_j\psi_{inc}$$

But

$$\int_\Gamma d\vec r_0 \int_{\vec n\vec\Omega_0<0} |\vec n\vec\Omega_0|\,\mathcal{S}_n(\vec r_0,\vec\Omega_0)\,L_i^* G = \int_\Gamma d\vec r_0 \int_{\vec n\vec\Omega_0<0} |\vec n\vec\Omega_0|\,\mathcal{S}_n(\vec r_0,\vec\Omega_0) \int_\Gamma d\vec r \int_{\vec n\vec\Omega>0} |\vec n\vec\Omega|\,\mathcal{S}_i^*(\vec r,\vec\Omega)$$
$$\times\, G(\vec r,\vec\Omega|\vec r_0,\vec\Omega_0)\,d\vec\Omega\,d\vec\Omega_0$$

$$= K_m^* L_i^* G$$

Let
$$\boxed{\begin{aligned} P_m^* f &= \int_D d\bar{r} \int_{4\pi} d\bar{\Omega}\, \mathcal{S}_m(\bar{r},\bar{\Omega})\, f(\bar{r},\bar{\Omega}) \\ P_m f &= \int_D d\bar{r} \int_{4\pi} d\bar{\Omega}\, \mathcal{S}_m^*(\bar{r},\bar{\Omega})\, f(\bar{r},\bar{\Omega}) \end{aligned}}$$

$$P_m Q = \sum q_n \int_D d\bar{r} \int_{4\pi} d\bar{\Omega}\, \mathcal{S}_m^*(\bar{r},\bar{\Omega}) \mathcal{S}_n(\bar{r},\bar{\Omega})$$
$$= \sum_n B_{mn} q_n$$
$$q_n = \sum_m (B^{-1})_{nm} P_m Q \stackrel{\Delta}{=} \sum B^{-1}_{nm} Q_m$$

$$L_k^* \phi = \int d\bar{r}_0 \int d\bar{\Omega}_0 \sum_n \mathcal{S}_n(\bar{r}_0,\bar{\Omega}_0)\, L_k^* G(\bar{r},\bar{\Omega}|\bar{r}_0,\bar{\Omega}_0) \sum_m (B^{-1})_{nm} P_m Q$$
$$+ \sum_n \left(L_k^* K_n^* G\right) \sum_j (C^{-1})_{nj} L_j \psi_{inc}$$

The relation ( ) is approximately true if we replace G by $\hat{G}$. Then:
$$L_k^{\alpha *} K_n \hat{G} = L_k^{\alpha *} K_n G_\infty - \sum_i \sum_j \left(K_n L_i G_\infty\right)\left(L_i K_j G_\infty\right)^+ \left(L_k^{\alpha *} K_j G_\infty\right)$$

with $\quad L_k^{*\alpha} \phi = \int_{\Gamma_\alpha} d\bar{r}_S \int_{\bar{n}\cdot\bar{\Omega}>0} |\bar{n}\cdot\bar{\Omega}|\, \mathcal{S}_k^*(\bar{r}_S,\bar{\Omega}) \cdots d\bar{\Omega}$

and $\quad L_k^* = \sum_\alpha L_k^{*\alpha} \quad , \quad \Gamma = \bigcup_\alpha \Gamma_\alpha$

let : $L_k^{*\alpha} \phi = J^+_{k,\alpha} \quad ; \quad L_i^\alpha \phi = J^-_{i,\alpha}$

$$J^+_{k,\alpha} = \sum_{n,m} \left(P_n^* L_k^{*\alpha} G\right) B^{-1}_{nm} Q_m + \sum_{n,j} \left(L_k^{*\alpha} K_n^* G\right) (C^{-1})_{nj} \sum_\beta J^-_{j,\beta}$$
$$P_n^* L_k^{*\alpha} \hat{G} = L_k^{\alpha *} P_n^* G_\infty - \sum_i \sum_j \left(P_n^* L_i G_\infty\right)\left(L_i K_j G_\infty\right)^+ \left(L_k^{\alpha *} K_j G_\infty\right)$$

$$\boxed{\begin{aligned} J^+_{k,\alpha} = &\sum_{n,m} \left\{ \left(L_k^{*\alpha} P_n^* G_\infty\right) - \sum_{i,j} \left(P_n^* L_i G_\infty\right)\left(L_i K_j G_\infty\right)^+ \left(L_k^{\alpha *} K_j G_\infty\right) \right\} B^{-1}_{nm} Q_m \\ + &\sum_{n,m} \left\{ \left(L_k^{\alpha *} K_n^* G_\infty\right) - \sum_{i,j} \left(K_n^* L_i G_\infty\right)\left(L_i K_j G_\infty\right)^+ \left(L_k^{\alpha *} K_j G_\infty\right) \right\} C^{-1}_{nj} \sum_\beta J^-_{j,\beta} \end{aligned}}$$

$$J^{+}_{k,\alpha} = \sum_{n,m} \left\{ \left(L_k^{*\alpha} P_n^{*} G_\infty \right) - \sum_{i,j} \left(L_k^{\alpha *} K_j G_\infty \right) \left(K_j L_i G_\infty \right)^{+} \left(L_i P_n^{*} G_\infty \right) \right\} B^{-1}_{nm} Q_m$$

$$+ \sum_{n,m} \left\{ L_k^{*\alpha} K_n^{*} G_\infty \right) - \sum_{i,j} \left(L_k^{\alpha *} K_j G_\infty \right) \left(K_j L_i G_\infty \right)^{+} \left(L_i K_n^{*} G_\infty \right) \right\} C^{-1}_{nm} \sum_{\beta} J^{-}_{m,\beta}$$

a relation which corresponds to the response matrix formalism :

$$\boxed{J^{+}_{k,\alpha} = \sum_{\beta,m} R^{SS}_{k,m}(\alpha,\beta)\, J^{-}_{m,\beta} + \sum_{m} R^{SV}_{k,m}(\alpha)\, Q_m}$$

In order to evaluate R^{SS}, R^{SV} we ~~must~~ either use the exact G_∞ or an approximate form. If $\mathcal{S}_n(\vec r,\vec\Omega)$ is a solution of the homogeneous B. equation

$$\tilde G_\infty(\vec r,\vec\Omega\,|\,\vec r_0,\vec\Omega_0) = \sum_n \sum_m \mathcal{S}_n(\vec r,\vec\Omega)\, A^{-1}_{nm}\, \mathcal{S}^{*}_m(\vec r_0,\vec\Omega_0)$$

with $\quad A_{m,n} = C_{m,n}$

1°/ $L_k^{*\alpha} P_n^{*} G_\infty = \sum_{p,q} L_k^{*\alpha} \mathcal{S}_p(\vec r,\vec\Omega)\, P_n^{*} \mathcal{S}_q^{*}(\vec r_0,\vec\Omega_0)\, A^{-1}_{pq}$

$$= \sum_{p,q} \int_{\Gamma_\alpha} d\vec r_s \int_{\vec n\vec\Omega>0} |\vec n\vec\Omega|\, \mathcal{S}^{*}_k(\vec r_s,\vec\Omega)\, \mathcal{S}_p(\vec r_s,\vec\Omega)\, d\vec\Omega$$

$$\times \int_{\mathcal{D}} d\vec r \int \mathcal{S}_n(\vec r_0,\vec\Omega_0)\, \mathcal{S}_q^{*}(\vec r_0,\vec\Omega_0)\, d\vec\Omega_0\; C^{-1}_{pq}$$

$$= \sum_{p,q} C^{\alpha}_{kp}\, C^{-1}_{pq}\, B_{qn} = \left(C^{\alpha} C^{-1} B \right)_{kn}$$

2°/ $L_k^{*\alpha} K_n^{*} G_\infty = \sum_{p,q} L_k^{*\alpha} \mathcal{S}_p(\vec r,\vec\Omega)\, K_n^{*} \mathcal{S}_q^{*}(\vec r_0,\vec\Omega_0)\, A^{-1}_{pq}$

$$= \sum_{p,q} \int_{\Gamma_\alpha} d\vec r_s \int_{\vec n\vec\Omega>0} |\vec n\vec\Omega|\, \mathcal{S}^{*}_k(\vec r_s,\vec\Omega)\, \mathcal{S}_p(\vec r_s,\vec\Omega)\, d\Omega \times \int_{\Gamma} d\vec r \int_{\vec n\vec\Omega_0 \cancel{<0}} |\vec n\vec\Omega_0|\, d\vec\Omega_0$$

$$\times\; \mathcal{S}^{*}_n(\vec r_0,\vec\Omega_0)\, \mathcal{S}_q(\vec r_0,\vec\Omega_0)\, A^{-1}_{pq}$$

$= \sum_{pq} C^{\alpha}_{kp}\, C^{-1}_{pq}\, E_{qn}$ au lieu de $\sum C^{\alpha}_{kq}\, C^{-1}_{pq}\, C_{qn}$ attendu

$$E_{pn} = \int_\Gamma d\bar{r} \int_{\bar{n}\bar{\Omega}_0<0} |n\Omega_0| \varphi_k(r_0,\Omega_0) \mathcal{J}_n(\bar{r}_0,\Omega_0)\, d\Omega_0$$

3°/ $L_k^{\alpha *} K_j G_\infty = (C^\alpha C^{-1} D)_{kj}$.

4°/ $K_j L_i G_\infty = \sum_{p,q} \int_\Gamma d\bar{r}_0 \int_{\bar{n}\bar{\Omega}_0>0} |\bar{n}\bar{\Omega}_0| \varphi_j^*(\bar{r}_0,-\bar{\Omega}_0) \varphi_q^*(\bar{r}_0,\bar{\Omega}_0)\, A_{pq}^{-1}$

$$\int d\bar{r} \int_{\bar{n}\bar{\Omega}<0} |\bar{n}\bar{\Omega}|\, \varphi_i^*(\bar{r},\bar{\Omega})\, \varphi_p(\bar{r},\bar{\Omega})$$

$$= \sum_{p,q} D_{qj}\, A_{pq}^{-1}\, C_{ip} = (CA^{-1}D)_{ij}.$$

5°/ $L_i P_n^* G_\infty = \sum_{p,q} \int_\Gamma d\bar{r}_0 \int_{\bar{n}\bar{\Omega}<0} |\bar{n}\bar{\Omega}|\, \varphi_i^*(\bar{r},\bar{\Omega})\, \varphi_p(\bar{r},\bar{\Omega})\, d\bar{\Omega}\, A_{pq}^{-1}$

$$\times \int_D d\bar{r} \int d\bar{\Omega}\, \varphi_n(\bar{r},\bar{\Omega})\, \varphi_q^*(\bar{r}_0,\bar{\Omega}_0).$$

$$= \sum C_{ip}\, A_{pq}^{-1}\, B_{qn} = (CA^{-1}B)_{in}$$

6°/ $L_i K_n^* G_\infty = \sum_{p,q} \int_\Gamma d\bar{r}_0 \int_{\bar{n}\bar{\Omega}<0} |\bar{n}\bar{\Omega}|\, \varphi_i^*(\bar{r},\bar{\Omega})\, \varphi_p(\bar{r},\bar{\Omega})\, d\bar{\Omega}\, A_{pq}^{-1}$

$$\int_\Gamma d\bar{r}_0 \int_{\bar{n}\bar{\Omega}_0 \cancel{>} 0} \varphi_n^*(\bar{r}_0,-\bar{\Omega}_0)\, \varphi_q^*(\bar{r}_0,\bar{\Omega}_0)\, |\bar{n}\bar{\Omega}_0|\, d\bar{\Omega}_0$$

$$= \sum_{p,q} C_{ip}\, A_{pq}^{-1}\, E_{qn} = (CA^{-1}E)_{in}$$

Si $A \neq C$

$$J_{k,\alpha}^+ = \sum_m (C^\alpha A^{-1} B \cancel{B^{-1}})_{km} Q_m - \sum \left[(C^\alpha A^{-1} D)(CA^{-1}D)^+(CA^{-1})\right]_{km} Q_m$$

$$+ \sum_m (C^\alpha A^{-1} E C^{-1})_{km} J_m^- - \sum_m \left[(C^\alpha A^{-1} D)(CA^{-1}D)^+(CA^{-1}EC^{-1})\right]_{km} J_m^-$$

Si $A = C$

$$J_{k,\alpha}^+ = \sum_m \left[(C^\alpha C^{-1})_{km} - \left((C^\alpha C^{-1} D) D^+\right]_{km} Q_m\right.$$

$$+ \sum_m \left\{ [C^\alpha C^{-1} E C^{-1}]_{km} - [C^\alpha C^{-1} D D^+ E C^{-1}]_{km} \right\} J_m^{-}$$

$A \neq C$

$$J^+_{k,\alpha} = \sum_m \Big[C^\alpha A^{-1} - (C^\alpha A^{-1} C^T)(C A^{-1} C^T)^+ (C A^{-1}) \Big]_{km} Q_m$$

$$+ \sum_m \Big[C^\alpha A^{-1} - (C^\alpha A^{-1} C^T)(C A^{-1} C^T)^+ (C A^{-1}) \Big] \Big[E^T C^{-1} \Big]_{km} J^-_m$$

$$J^+_{k,\alpha} = \sum_{n,m} \Big[\big(L_k^{*\alpha} P_n^* G_\infty\big) - \sum_{i,j} \big(L_k^{*\alpha} K_j G_\infty\big)\big(K_j L_i G_\infty\big)^+ \big(L_i P_n^* G_\infty\big) \Big] B^{-1}_{nm} Q_m$$

$$+ \sum_{n,m} \Big[\big(L_k^{*\alpha} K_n^* G_\infty\big) - \sum_{i,j} \big(L_k^{*\alpha} K_j G_\infty\big)\big(K_j L_i G_\infty\big)^+ \big(L_i K_n^* G_\infty\big) \Big] C^{-1}_{nm} J^-_m$$

$$J^+_{k,\alpha} = \sum L_k^{*\alpha} \Big[I - K_j G_\infty (K_j L_i G_\infty)^+ L_i \Big] P_n^* G_\infty B^{-1}_{nm} Q_m$$

$$+ \sum L_k^{*\alpha} \Big[I - K_j G_\infty (K_j L_i G_\infty)^+ L_i \Big] K_n^* G_\infty C^{-1}_{nm} J^-_m$$

$$L_k K_\ell \hat{G} = L_k K_\ell G_\infty - \sum_{i,j} (K_\ell L_i G_\infty)(L_i K_j G_\infty)^+ (L_k K_j G_\infty)$$

$$= L_k K_\ell G_\infty - \sum_{i,j} (L_k K_j G_\infty)(K_j L_i G_\infty)^+ (L_i K_\ell G_\infty)$$

$$= 0$$

La raison du $0=0$ est le fait que $\hat{G}_\infty = \sum \varphi_n(\vec r, \vec\Omega)\, A^{-1}_{nm}\, \varphi^\times_m(\vec r_0, \vec\Omega_0)$
c.a.d une somme de produits de fonctions de $(\vec r, \vec\Omega)$ et $(\vec r_0, \vec\Omega_0)$ et que

$$\hat{G} = G_\infty - T(G_\infty)$$

or $T(f) = f$ pour f un produit tensoriel